



SEQUENCES AND SERIES

REVISION WORKBOOK



Sequences and series

Quick review

A **sequence** is an ordered list $u_1, u_2, u_3, \dots$ with n th term u_n ; a **series** adds its terms, so $S_n = u_1 + u_2 + \dots + u_n$.

Arithmetic sequences and series

Identifier: a constant difference $+d$ between consecutive terms, so $d = u_{n+1} - u_n$.

KEY VARIABLES

u_1 first term u_n the n th term
 d common difference S_n sum of first n terms
 n term number

GENERAL TERM (u_n)

$$u_n = u_1 + (n - 1)d$$

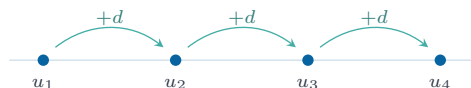
SUM OF n TERMS (S_n)

$$S_n = \frac{n}{2}(2u_1 + (n - 1)d)$$

ALT SUM, WHEN u_n IS KNOWN

$$S_n = \frac{n}{2}(u_1 + u_n)$$

MOVING ALONG THE SEQUENCE



Each step adds one d , so jumping k terms adds kd .

EXAM TIP

Jump trick. Move between any two known terms directly:

$$u_n = u_m + (n - m)d$$

E.g. $u_7 = u_3 + 4d$, since the gap is $7 - 3 = 4$. No need to find u_1 first.

WORKED EXAMPLE

$u_3 = 11$ and $u_7 = 27$. Find u_{15} and S_{15} .

Step 1 The gap from u_3 to u_7 is 4, so jump forward $4d$

$$27 = 11 + 4d \Rightarrow d = 4$$

Step 2 Step back 2 terms from u_3 to reach u_1

$$u_1 = u_3 - 2d = 11 - 8 = 3$$

Step 3 Step forward 12 terms from u_3 to reach u_{15}

$$u_{15} = u_3 + 12d = 11 + 48 = 59$$

Step 4 Both ends known, so use the alt sum

$$S_{15} = \frac{15}{2}(3 + 59) = 465$$

Geometric sequences, series and the sum to infinity

Identifier: a constant ratio $\times r$ between consecutive terms, so $r = u_{n+1} \div u_n$.

GENERAL TERM (u_n)

$$u_n = u_1 r^{n-1}$$

COMMON RATIO

$$r = \frac{u_{n+1}}{u_n}$$

SUM OF n TERMS (S_n), $r \neq 1$

$$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}$$

Left form when $r > 1$, right form when $r < 1$.

SUM TO INFINITY (S_∞)

$$S_\infty = \frac{u_1}{1 - r}, \quad |r| < 1$$

Converges: $|r| < 1$ **Diverges:** $|r| \geq 1$

Only a converging series has a sum to infinity: the terms shrink towards zero, so adding every term still settles on one finite value.

EXAM TIP

Jump trick. $u_n = u_m r^{n-m}$, e.g. $u_7 = u_3 r^4$.

Inequalities in n . Graph S_n and $y = k$ and find the crossing. Since n is a positive integer, test the integers either side and keep the one that fits.

WORKED EXAMPLE

$u_2 = 6$ and $u_5 = 48$. Find r , u_{10} and S_{10} .

Step 1 The gap from u_2 to u_5 is 3, so multiply by r^3

$$48 = 6r^3 \Rightarrow r^3 = 8 \Rightarrow r = 2$$

Step 2 Step forward 5 terms from u_5 to reach u_{10}

$$u_{10} = u_5 r^5 = 48 \times 32 = 1536$$

Step 3 Step back one term to reach u_1

$$u_1 = u_2 \div r = 6 \div 2 = 3$$

Step 4 Now sum, with $r > 1$ so use the left form

$$S_{10} = \frac{3(2^{10} - 1)}{2 - 1} = 3069$$

Questions

LEVEL 1 • • • Foundational

1. [Maximum mark: 6] • **Paper 1**, calculator not allowed

Consider an arithmetic sequence where $u_1 = 5$ and $u_4 = 17$.

- (a) Find the common difference. [2]
- (b) Find u_{10} . [2]
- (c) Find S_{10} . [2]

2. [Maximum mark: 6] • **Paper 1**, calculator not allowed

Consider a geometric sequence where $u_1 = 24$ and $u_4 = 3$.

- (a) Show that $r = \frac{1}{2}$. [2]
- (b) Find u_7 , giving an exact answer. [2]
- (c) Find S_{∞} . [2]

3. [Maximum mark: 5] • **Paper 2**, calculator allowed

Consider the arithmetic sequence 7, 11, 15, 19, ...

- (a) Write down the common difference. [1]
- (b) Find u_{25} . [2]
- (c) Find $\sum_{n=1}^{25} u_n$. [2]

4. [Maximum mark: 6] • **Paper 2**, calculator allowed

The population of a town is currently 24 000 and is increasing by 3.5% each year.

- (a) Find the population of the town after 10 years. [3]
- (b) Find the number of complete years it takes for the population to first exceed 40 000. [3]

LEVEL 2 ••• Standard

5. [Maximum mark: 8] • Paper 1, calculator not allowed

The first three terms of an arithmetic sequence are $k - 3$, $2k + 1$ and $4k - 2$.

(a) Find the value of k . [3]

(b) Hence write down u_1 and the common difference. [2]

(c) Find $\sum_{n=1}^{20} u_n$. [3]

6. [Maximum mark: 8] • Paper 1, calculator not allowed

An infinite geometric sequence has first term $u_1 = 18$ and sum to infinity $S_\infty = 27$.

(a) Find the common ratio. [3]

(b) Find u_5 , giving an exact answer. [2]

(c) Find S_4 , giving an exact answer. [3]

7. [Maximum mark: 7] • **Paper 2**, calculator allowed

A theatre has 20 seats in the first row. Each row after the first has 2 more seats than the row in front of it.

- (a) Find the number of seats in the 15th row. [2]
- (b) Find the total number of seats in the first 15 rows. [2]
- (c) The theatre has 1032 seats in total. Find the number of rows in the theatre. [3]

8. [Maximum mark: 6] • **Paper 2**, calculator allowed

A machine is bought for \$85 000. Its value depreciates by 12% each year.

- (a) Find the value of the machine after 4 years, giving your answer correct to the nearest dollar. [3]

- (b) Find the number of complete years after which the value of the machine first falls below \$20 000. [3]

LEVEL 3 ••• Challenging

9. [Maximum mark: 7] • **Paper 1**, calculator not allowed

The sum to infinity of a geometric sequence is 32, and the sum of its first two terms is 24.

- (a) Find the two possible values of the common ratio. [4]

- (b) Find the value of u_1 corresponding to each value of r . [3]

10. [Maximum mark: 8] • **Paper 1**, calculator not allowed

Consider the sequence

$$\ln x^{27}, \ln x^9, \ln x^3, \dots \quad \text{where } x > 1.$$

- (a) Show that the sequence is geometric, and state the common ratio. [3]
- (b) Find S_∞ , giving your answer in the form $p \ln x$ where $p \in \mathbb{Q}$. [3]
- (c) Given that $S_\infty = 81$, find the value of x . [2]

11. [Maximum mark: 18] • Paper 2, Section B, calculator allowed

Anya and Ben each begin a new job on the same day.

Anya's salary in her first year is \$32 000. Her salary increases by \$1800 each year.

- (a) Find Anya's salary in her 10th year. [2]
- (b) Find the total amount Anya earns in her first 10 years. [2]

Ben's salary in his first year is \$30 000. His salary increases by 5% each year.

- (c) Find Ben's salary in his 10th year, giving your answer correct to the nearest dollar. [2]
- (d) Find the total amount Ben earns in his first 10 years, giving your answer correct to the nearest dollar. [3]

(This question continues on the following page)

Although Ben's salary is lower than Anya's in the first year, it eventually becomes greater.

- Carla begins the same job on the same day. Her annual salaries also form an arithmetic sequence. It is known that:

- her salary in her 5th year is \$38 400
- the total amount she earns in her first 12 years is \$498 600

- (f) Find Carla's salary in her first year. [5]

12. [Maximum mark: 23] • **Paper 3 (HL)**, calculator allowed

This question combines sequences and series with sigma notation, proof by mathematical induction and differentiation.

This question asks you to investigate sums of the form $\sum r \cdot x^r$ and to find a general result.

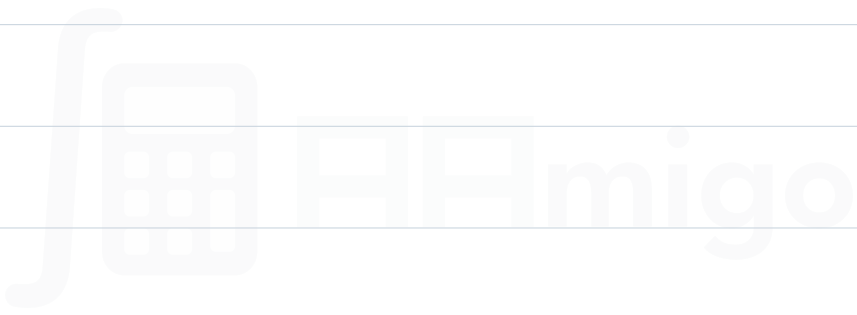
Consider the series

$$S_n = \sum_{r=1}^n r \cdot 2^r.$$

- (a) Find the value of S_1 , S_2 , S_3 and S_4 . [3]

It can be shown that S_n may be written in the form $S_n = (an + b)2^{n+1} + c$, where $a, b, c \in \mathbb{Z}$.

- (b) (i) Use your answers to part (a) to find the values of a , b and c . [4]
(ii) Verify that this expression gives the correct value of S_5 . [2]



(This question continues on the following page)

(c) Prove by mathematical induction that $S_n = (n-1)2^{n+1} + 2$ for all $n \in \mathbb{Z}^+$. [7]

(d) Hence find $\sum_{r=11}^{20} r \cdot 2^r$. [3]

$$G_n(x) = \sum_{r=0}^n x^r.$$

- (e) (i) Write down an expression for $G_n(x)$ in terms of x and n . [2]
- (ii) Show that $U_n(x) = x G'_n(x)$. [2]

Mark schemes

IN THE EXAM

Mark codes work the same way here as in a real IB mark scheme. **M** is a method mark, awarded for a correct approach even if the arithmetic then slips. **A** is an accuracy mark, awarded for a correct value, and it depends on the method mark before it. **R** is awarded for a reason or justification. **AG** means the answer is given in the question, so your working must actually reach it. **FT** means a later answer is marked through from your own earlier value. Brackets, as in **(M1)**, mean the mark can be implied by correct work that follows.

1. [Maximum mark: 6] • **Paper 1**, calculator not allowed

- (a) $u_4 = u_1 + 3d \Rightarrow 5 + 3d = 17$ (M1)
 $d = 4$ A1
 [2 marks]
- (b) $u_{10} = 5 + 9(4)$ (M1)
 $= 41$ A1
 [2 marks]
- (c) $S_{10} = \frac{10}{2}(2(5) + 9(4))$ or $\frac{10}{2}(5 + 41)$ (M1)
 $= 230$ A1
Note: FT from an incorrect d or u_{10} .
 [2 marks]

2. [Maximum mark: 6] • **Paper 1**, calculator not allowed

- (a) $u_4 = u_1 r^3 \Rightarrow 24r^3 = 3$ M1
 $r^3 = \frac{1}{8} \Rightarrow r = \frac{1}{2}$ AG
Note: The answer is given, so both lines must be seen.
 [2 marks]
- (b) $u_7 = 24 \left(\frac{1}{2}\right)^6$ (M1)
 $= \frac{3}{8}$ A1
Note: Award A0 for 0.375: the question asks for an exact answer.
 [2 marks]

(c) $S_{\infty} = \frac{24}{1 - \frac{1}{2}}$ (M1)

$= 48$ A1

[2 marks]

3. [Maximum mark: 5] • Paper 2, calculator allowed

(a) $d = 4$ A1

[1 mark]

(b) $u_{25} = 7 + 24(4)$ (M1)

$= 103$ A1

[2 marks]

(c) $S_{25} = \frac{25}{2}(7 + 103)$ or $\frac{25}{2}(2(7) + 24(4))$ (M1)

$= 1375$ A1

Note: The sigma notation is asking for S_{25} , not for u_{25} .

[2 marks]

4. [Maximum mark: 6] • Paper 2, calculator allowed

(a) geometric with $u_1 = 24\,000$ and $r = 1.035$ (M1)

$24\,000(1.035)^{10} = 33\,854.37\dots$ (A1)

$= 33\,900$ (3 s.f.) A1

Note: Using $u_1 r^{n-1}$ with $n = 10$ gives $32\,709\dots$ and scores A0: after 10 years the multiplier has been applied 10 times.

[3 marks]

(b) $24\,000(1.035)^n > 40\,000$ (M1)

$n > 14.848\dots$ (from a GDC table or graph) (A1)

15 years A1

Note: Accept a table showing $38\,848\dots$ at $n = 14$ and $40\,208\dots$ at $n = 15$.

[3 marks]

5. [Maximum mark: 8] • Paper 1, calculator not allowed

(a) $(2k + 1) - (k - 3) = (4k - 2) - (2k + 1)$ (equal differences) M1

$k + 4 = 2k - 3$ (A1)

$k = 7$ A1

[3 marks]

- (b) $u_1 = 4$ A1
 $d = 11$ A1

Note: The three terms are 4, 15 and 26.

[2 marks]

- (c) $S_{20} = \frac{20}{2}(2(4) + 19(11))$ (M1)(A1)
 $= 2170$ A1

Note: FT from an incorrect u_1 or d .

[3 marks]

6. [Maximum mark: 8] • **Paper 1**, calculator not allowed

- (a) $\frac{18}{1-r} = 27$ (M1)
 $1-r = \frac{18}{27} = \frac{2}{3}$ (A1)
 $r = \frac{1}{3}$ A1

[3 marks]

- (b) $u_5 = 18\left(\frac{1}{3}\right)^4$ (M1)
 $= \frac{2}{9}$ A1

[2 marks]

- (c) $S_4 = \frac{18\left(1 - \left(\frac{1}{3}\right)^4\right)}{1 - \frac{1}{3}}$ (M1)(A1)
 $= \frac{80}{3}$ A1

Note: Award A0 for 26.7 or $26\frac{2}{3}$: an exact answer is asked for, and a single fraction is expected.

[3 marks]

7. [Maximum mark: 7] • **Paper 2**, calculator allowed

- (a) $u_{15} = 20 + 14(2)$ (M1)
 $= 48$ seats A1

[2 marks]

- (b) $S_{15} = \frac{15}{2}(20 + 48)$ (M1)
 $= 510$ seats A1

[2 marks]

- (c) $\frac{n}{2}(2(20) + (n-1)2) = 1032$ (M1)
 $n^2 + 19n - 1032 = 0$ or equivalent solved on a GDC (A1)
 $n = 24$ rows A1
Note: The other root is $n = -43$, which is rejected because $n \in \mathbb{Z}^+$.

[3 marks]

8. [Maximum mark: 6] • Paper 2, calculator allowed

- (a) geometric with $r = 0.88$ (M1)
 $85\,000(0.88)^4 = 50\,974.1056$ (A1)
 $= \$50\,974$ A1
- (b) $85\,000(0.88)^n < 20\,000$ (M1)
 $n > 11.318\dots$ (A1)
12 years A1

Note: At $n = 11$ the value is $\$20\,831\dots$, which is still above $\$20\,000$.

[3 marks]

9. [Maximum mark: 7] • Paper 1, calculator not allowed

- (a) $\frac{u_1}{1-r} = 32$ and $u_1 + u_1r = 24$ (M1)
 $u_1 = 32(1-r) \Rightarrow 32(1-r)(1+r) = 24$ (M1)
 $1-r^2 = \frac{3}{4} \Rightarrow r^2 = \frac{1}{4}$ (A1)
 $r = \frac{1}{2}$ or $r = -\frac{1}{2}$ A1

Note: Both values satisfy $|r| < 1$, so neither is rejected.

[4 marks]

- (b) $r = \frac{1}{2} \Rightarrow u_1 = 32\left(\frac{1}{2}\right) = 16$ (M1)A1
 $r = -\frac{1}{2} \Rightarrow u_1 = 32\left(\frac{3}{2}\right) = 48$ A1

[3 marks]

10. [Maximum mark: 8] • Paper 1, calculator not allowed

- (a) $\ln x^{27} = 27 \ln x$, $\ln x^9 = 9 \ln x$, $\ln x^3 = 3 \ln x$ (M1)
 $\frac{9 \ln x}{27 \ln x} = \frac{3 \ln x}{9 \ln x} = \frac{1}{3}$, a constant ratio, so the sequence is geometric (R1)
 $r = \frac{1}{3}$ (A1)

Note: Since $x > 1$, $\ln x \neq 0$ and the division is valid.

[3 marks]

- (b) $\left| \frac{1}{3} \right| < 1$, so S_∞ exists (R1)
 $S_\infty = \frac{27 \ln x}{1 - \frac{1}{3}}$ (M1)
 $= \frac{81}{2} \ln x$ (A1)

[3 marks]

- (c) $\frac{81}{2} \ln x = 81 \Rightarrow \ln x = 2$ (M1)
 $x = e^2$ (A1)

Note: Award A0 for 7.39: an exact value is expected on Paper 1.

[2 marks]

11. [Maximum mark: 18] • **Paper 2, Section B**, calculator allowed

- (a) $u_{10} = 32\,000 + 9(1800)$ (M1)
 $= \$48\,200$ (A1)

[2 marks]

- (b) $S_{10} = \frac{10}{2}(32\,000 + 48\,200)$ (M1)
 $= \$401\,000$ (A1)

[2 marks]

- (c) $30\,000(1.05)^9 = 46\,539.84\dots$ (M1)
 $= \$46\,540$ (A1)

Note: The exponent is 9, not 10: the first year is u_1 .

[2 marks]

- (d) $S_{10} = \frac{30\,000(1.05^{10} - 1)}{1.05 - 1}$ (M1)(A1)
 $= 377\,336.77\dots = \$377\,337$ (A1)

[3 marks]

- (e) $30\,000(1.05)^{n-1} > 32\,000 + 1800(n-1)$ (M1)
 attempt to compare the two salaries for successive values of n , on a GDC table or by graphing both (M1)
 $n = 12$: Ben \$51 310, Anya \$51 800 (Ben is still behind) (A1)
 $n = 13$: Ben \$53 875, Anya \$53 600, so the 13th year A1

Note: An answer of 13 with no supporting comparison scores the final A1 only if a valid method is visible.

[4 marks]

- (f) $u_1 + 4d = 38\,400$ M1
 $\frac{12}{2}(2u_1 + 11d) = 498\,600$ M1
 $2u_1 + 11d = 83\,100$ (A1)
 $3d = 6300 \Rightarrow d = 2100$ A1
 $u_1 = 38\,400 - 4(2100) = \$30\,000$ A1

Note: Accept solving the two equations on a GDC. FT for a correct u_1 from an incorrect d .

[5 marks]

12. [Maximum mark: 23] • Paper 3 (HL), calculator allowed

- (a) $S_1 = 1 \cdot 2 = 2$, $S_2 = 2 + 2 \cdot 4 = 10$ A1
 $S_3 = 10 + 3 \cdot 8 = 34$ A1
 $S_4 = 34 + 4 \cdot 16 = 98$ A1

[3 marks]

- (b)(i) $n = 1$: $(a+b)2^2 + c = 2$; $n = 2$: $(2a+b)2^3 + c = 10$; $n = 3$: $(3a+b)2^4 + c = 34$ M1
 $a = 1$ A1
 $b = -1$ A1
 $c = 2$ A1

Note: Accept solving the three equations on a GDC. The expression is $S_n = (n-1)2^{n+1} + 2$.

[4 marks]

- (b)(ii) $(5-1)2^6 + 2 = 4(64) + 2 = 258$ M1
 $S_5 = 98 + 5 \cdot 32 = 258$, so the expression is verified A1

[2 marks]

(c) $n = 1$: LHS = $1 \cdot 2^1 = 2$, RHS = $(1 - 1)2^2 + 2 = 2$, so the statement is true for $n = 1$

A1

assume the statement is true for $n = k$, that is $\sum_{r=1}^k r \cdot 2^r = (k - 1)2^{k+1} + 2$

M1

$$\sum_{r=1}^{k+1} r \cdot 2^r = \sum_{r=1}^k r \cdot 2^r + (k + 1)2^{k+1}$$

M1

$$= (k - 1)2^{k+1} + 2 + (k + 1)2^{k+1}$$

A1

$$= 2^{k+1}((k - 1) + (k + 1)) + 2 = 2^{k+1}(2k) + 2$$

A1

$$= k 2^{k+2} + 2 = ((k + 1) - 1)2^{(k+1)+1} + 2, \text{ which is the statement for } n = k + 1$$

A1

the statement is true for $n = 1$, and truth for $n = k$ implies truth for $n = k + 1$, so by

R1

mathematical induction it is true for all $n \in \mathbb{Z}^+$

Note: The final R1 is awarded only if the base case and the inductive step have both been established.

[7 marks]

(d) $\sum_{r=11}^{20} r \cdot 2^r = S_{20} - S_{10}$

(M1)

$$S_{20} = 19 \cdot 2^{21} + 2 = 39\,845\,890, \quad S_{10} = 9 \cdot 2^{11} + 2 = 18\,434$$

(A1)

$$= 39\,827\,456$$

A1

Note: Subtracting S_{11} instead of S_{10} scores M0: the sum starts at $r = 11$, so the eleventh term is included.

[3 marks]

(e)(i) geometric series with first term 1 and ratio x , and $n + 1$ terms

(M1)

$$G_n(x) = \frac{x^{n+1} - 1}{x - 1}$$

A1

[2 marks]

(e)(ii) $G_n(x) = 1 + x + x^2 + \cdots + x^n \Rightarrow G'_n(x) = \sum_{r=1}^n r x^{r-1}$

M1

$$x G'_n(x) = \sum_{r=1}^n r x^r = U_n(x)$$

AG

Note: Differentiating the closed form and simplifying is an equally valid method.

[2 marks]